1. Let A and B be bounded subsets of real numbers. Suppose that $A + B = \{a + b : a \in A, b \in B\}$. (i) Prove that $\sup(A+B) = \sup(A) + \sup(B)$. (ii) If $C = \{a + b : a^2 < 2, b < 5\}$, then find $\sup(C)$.

   Solution: (i) The proof is easy. Note that $a + b \leq \sup(A) + \sup(B)$ for all $a \in A$, $b \in B$. Hence, $\sup(A + B) \leq \sup(A) + \sup(B)$. To get the other inequality, we have that $a + b \leq \sup(A + B)$. Keep a fixed. Then $a + \sup(B) \leq \sup(A + B)$. Now, vary a. We get the result. (ii) The above works for unbounded sets too. Take A to the interval $(-\sqrt{2}, \sqrt{2})$ and $B = (-\infty, 5)$. Hence the required result is $\sqrt{2} + 5$.

   $\square$

2. Consider the sequence (y_n) given by $y_{n+1} = \sqrt{1 + 2y_n}$ with $y_1 = 1$. Prove that (y_n) converges and find the limit.

   Solution: Note that the given sequence is strictly positive. Also, $y_n \geq 1$ for all n, and $y_n \leq 1 + \sqrt{2}$ which can be seen by induction on n. Moreover, by induction again on n, we see that the sequence is monotically increasing. Thus the limit exists and is given by the positive root of $x^2 = 1 + 2x$.

   $\square$

3. Decide with justification whether the following are true. (i) Let (x_n) be a bounded sequence of real numbers M be the supremum of (x_n). Then there exists a subsequence (x_{n_k}) converging to M. (ii) There is no ordering of the complex field $\mathbb{C}$ such that it becomes an ordered complex filed. (iii) Let (x_n) and (y_n) be Cauchy sequences and $y_n > 0$ for all n. Then (x_n/y_n) is Cauchy.

   Solution: (i) This is not true; e.g. $1, 1/2, 1/3, \ldots.$. Every subsequence goes to 0, whereas the supremum is 1. (ii) The ordering should be compatible with the field operations. So, assume that $i > 0$. Then $i^2 > 0$, ie. $-1 > 0$. This together with $i > 0$ gives $-i > 0$. This is a contradiction. (iii) This is not true, eg, $x_n = 1$, $y_n = 1/n$, $n \geq 1$. Then $x_n/y_n = n$ goes to ∞. Hence it cannot be Cauchy. $\square$